Three-dimensional manipulatives for undergraduate geometry classes

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Abstract

This article will cover the use of some basic three-dimensional manipulatives, such as balloons and paper polyhedra, which can enhance undergraduate exploration of non-euclidean geometry. Students' intuition of non-euclidean spaces is strengthened, and their enjoyment of the subject is augmented by teamwork in a laboratory-style setting. Copies of sample laboratories and manipulatives are available through the author's website or by email from the author.

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1. Introduction

Students in my Spring 2004 undergraduate geometry course, Modern Geometry I, were having trouble absorbing geometric concepts in a traditionally-styled lecture. Most of these students were math or computer science majors who had taken calculus, and many of them were also secondary education majors. The first examination results demonstrated a lack of learning and a discussion with the students revealed a deepening dislike for the subject area, so I decided to take a more hands-on approach for teaching this course and started giving them group work in the form of laboratories relying heavily on three-dimensional manipulatives. The second examination results and the students' feedback both indicated that this approach was much more effective at cultivating both enjoyment of and deeper understanding of the subject matter. The sections which follow summarize the various laboratories which were completed by the students.

I highly recommend experimenting with the laboratories in addition to or in lieu of reading this document for those interested in learning more about non-euclidean geometry. This document is intended for those interested in collecting ideas for teaching the topic but fails to reflect the curiosity and exploratory tone of the laboratory environment.

2. Triangles on an elliptic surface

The objectives were:
1) Demonstrate to the students that the sum of angles on a sphere is more than 180 degrees.
2) Get students thinking about what a straight line is.
3) Convince students that areas of the triangles on the sphere have something to do with the sum of the degrees.

Students were each individually given a length and two angular measurements and were told to draw the appropriate triangle on a balloon using a ruler and a protractor. Rulers from Lenart spheres were used as a guide to help make all the balloons about the same size, and students were asked to draw the triangle on the most sphere-like portion of the balloon. As an interesting note, they all drew geodesics instinctively, without any discussion about what geodesics were. They were then told to estimate the area of their triangle using the familiar plane formula. Answers for all students were tabulated on the board.

This laboratory took one class hour, and some assistance was given. The results of the laboratory were encouraging, even though the plane formula for the triangle's area is rather inaccurate for many triangles on the sphere. Getting accurate measurements on a balloon also proved to be tricky, but the students were mature enough to understand the drawbacks of the tools being used.

3. Areas of triangles on an elliptic surface

The objectives were:
1) Students should recall or learn that the area of the sphere is $4\pi R^2$.
2) Students learn that a triangle on a sphere arises from three great circles.
3) Derive formula for area of triangle on sphere.
4) Generalize formula for area of triangle on sphere to area of any polygon on sphere.
5) Remove dependence on number of sides in polygon from formula for area of polygon on sphere.

Students were again given balloons to work with, pretending the balloons to be perfect spheres. They were asked to first draw two different great circles on their balloons and find the proportion of the area of the whole sphere contained in each of the lunes formed. The eight resulting angles were labeled as *a* or *a'*, with *a* and *a'* being supplementary angles. Next, they were walked through a proof of the formula for the surface area of the sphere. (Most of the students had difficulty with this part of the laboratory.) This picture shows a styrofoam model of the eight triangles the students drew on their balloons.

Next, a third great circle was drawn so as to split each of the four lunes into two triangles. Resulting angles were labeled as *b, b', c,* and *c',* with *b* and *b'* (and *c* and *c'*) being supplements and so that one triangle would have the angles *a, b,* and *c*. The eight resulting areas have for different area measures, so the sections were labeled *A, B,* C, and *D* so that areas of same area had the same label, and the triangle with the angles *a, b,* and *c* was labeled *A*. (Different labels will be suggested next time, as students had quite a bit of trouble distinguishing the small lowercase letters from one another as the balloon was rotated. A lesser amount of frustration resulted from the use of the capital letters matching the lowercase letters.) Now, knowing how to calculate the surface area of a lune and the surface area of a hemisphere allowed for algebraic manipulations to solve for *A* in terms of *a, b,* and *c*. (*A* and *B* , *A* and *C*, and *A* and *D* each form lunes.) Students found that $A = 4\,(a + b + c - \pi)\,R^2$ where *R* is the radius of the sphere.

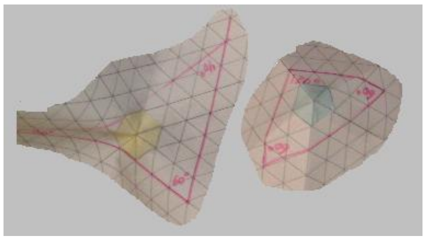
This result for the area of triangles on a sphere was extended to find the area of any convex polygon on a sphere, using induction. Finally, dependence upon the number of sides in the polygon was removed by measuring the supplements of the internal angles (called *turning angles* or *angles of deflection*) rather than the internal angles. This final formulation could be slightly altered to achieve a continuous representation of this result.

This laboratory took about two to three class hours and required much more assistance than the first laboratory. The results of this laboratory were good, but it would have been more effective to have it split into two laboratories rather than one two or three class hour laboratory. Students sometimes became discouraged with their lack of progress on a laboratory that would span over two or three class hours. After the first time this occurred, however, they did not mind as much.

4. Cone points

The objectives were:
1) Students create models of hyperbolic and elliptic cone points.
2) Students learn that adding (removing) angles at the cone points causes the triangle's internal angle sum to decrease (increase) by the amount of the added (removed) angle.

Students were shown samples of elliptic and hyperbolic cone points made from paper with equilateral triangular gridlines, and they were given materials to make their own. There was a classroom discussion about what was or was not developable, and a working definition was given that "a developable surface can be made out of paper". Students drew triangles (three intersecting geodesics) on the models and found the sum of the angles in them. They realized that any angles which they added to their models to create hyperbolic cone points would decrease the sum of the triangle's interior angles, but that any angles they removed from their models to create elliptic cone points would increase the sum of the triangle's interior angles.

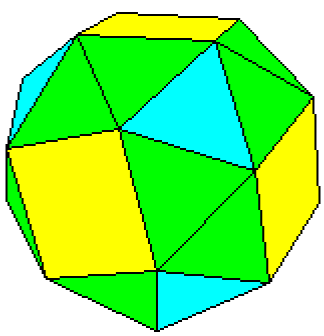


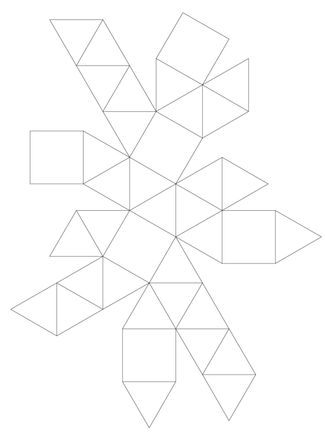
This laboratory was completed in one class hour with some assistance. Results of this laboratory were as expected, and the students enjoyed the exercise.

5. Total Gaussian curvature on various polyhedra

The objectives were:
1) Students learn that the sum of the angle deficits for a convex polyhedron is 4π.
2) Students learn that even for an object with hyperbolic cone points, the angle deficits formed by the introducing the hyperbolic cone points are balanced out by the elliptic cone points which also must arise.

Various regular convex paper polyhedra such as a tetrahedron, cube, trunctetrahedron, snubcube (shown below), and one paper bathtub-like shape were given to the students pre-assembled as well as in flat pattern format (as shown below). Students were asked to calculate the angle deficit for each cone point in each object. They added up the angle deficit for each object and found that they all totaled 4π radians.





This laboratory was completed in one class hour with some assistance. Results of this laboratory were as expected, and the students enjoyed the exercise. Some students asked for a proof of the fact that this type of object would always have an angle deficit of 4π radians, indicating that they were ready to be presented with the proof!

6. Proof of total curvature for polyhedra

The objectives were:
1) Learn about Euler characteristics.

2) Prove that the total Gauss curvature of a topologically spherical object composed of polygons is 4 π .

A model of a 'square donut' was provided, that is, a cube with a square 'tunnel' through it. The two outer faces with missing squares were split in half along the diagonal so that the object had 12 faces, each topologically equivalent to a disc. Students found that the total Gaussian curvature for this object was zero.

Next, the Euler characteristic, or $\chi = v + f - e$ (where *v* represents the number of vertices in the object, *f* represents the number of faces in the object, and *e* represents the number of edges in the object) was considered for triangular tessellations of surfaces of topologically spherical objects. The proof started with a planar case, where the Euler characteristic is one and moved to the spherical case (where it is two) by adding one additional face. (No proof was investigated for objects topologically equivalent to a torus.) The Euler characteristic, the total radians in a planar triangle, and some additional clues were given to the students, who were then able to arrive at the result that for any topologically spherical object composed of polygons, the total Gauss curvature is 4 π , which is 'coincidentally' also the surface area of a unit sphere.

This laboratory was completed in two to three class hours with considerable assistance. The results of this laboratory varied greatly by student. Many students understood the entire laboratory completely, but many students were (unfortunately) satisfied to believe the results and borrow work from their classmates.

7. Gauss-Bonnet Theorem

The objectives were:
1) Learn that the sum of angle deficits within a curve on a polyhedron and angle defects on the curve is 2 π (Gauss-Bonnet theorem).
2) Cause students to suspect that this result holds for non-polyhedral surfaces as well.

First, students were asked to draw a straight line on a piece of paper and observe the line as the paper was distorted into a cone or cylinder or other arrangement. Next, photographs of an icosidodecahedron with closed curves drawn on it were provided in the laboratory (for which the model was available at the front of the room). In the first photograph, the closed curve encircled only one elliptic cone point, for which the students calculated the angle deficit. For regular polyhedra, it is convenient to create closed curves by connecting the centers of adjacent faces. The closed curves so created will often cross an edge between faces perpendicularly to the edge, making it easy to conclude that the path is indeed a geodesic. Students were able to figure out the total turning angle for the closed curve on the icosidodecahedron. The second photograph showed a triangle drawn on an icosidodecahedron. The internal angles for that triangle were labeled for the students. From this, they were able to find the total turning angle for the curve, and deduce the angle deficit at each of the three identical cone points enclosed by the curve. This result matched their previous results for the same polyhedron. A third photograph gave a closed pentagon-like curve on a truncated icosahedron. Analogous values were calculated for this shape as for the icosidodecahedron.

Students were given photos of three objects (a sphere, a cylinder, and a star-like object) each encircled with curves (ribbons) lying flat along their surfaces. They correctly concluded that the total Gauss curvature enclosed by either side of the curve must be 2 π since each curve experienced no angular deflection. Details on the geometry of the star-like object were given to students so they could verify the result using techniques from previous laboratories. Finally, students were given diagrams representing surfaces and were asked to use algebra with the Gauss-Bonnet theorem to find missing information such as the Gauss curvature at a single point.

This laboratory was completed in two to three class hours with considerable assistance. The results of this laboratory were mostly positive, but some students could have used more time on the laboratory.

8. Examination

The examination for this sequence of laboratory exercises relied heavily on the use of digital photographs, some of which were taken directly from the laboratory exercises. Students were given a review sheet which highlighted some of the concepts they would be responsible for. The examination emphasized later material more than earlier material and did not emphasize supplemental material, such as the proof for the surface area of the sphere. Objects which appeared in the examination as photographs were available at the front of the room for students to borrow during the examination.

9. Conclusions

After reviewing the students' examination results, reflecting on their verbal responses, and considering their overall attitudes about geometry, I have to conclude that the course was much more successful after introducing hands-on laboratories. Another important change which contributed to the improved results was that I decided to cover less material but cover material more deeply, so the students would have the chance to think over more important concepts for longer periods of time. It seems that even with a decreased emphasis on covering material, more material is being absorbed. The most important change to report is that my students literally went from 'hating' geometry to saying that they looked forward to the class and were enjoying the material.

Acknowledgements

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